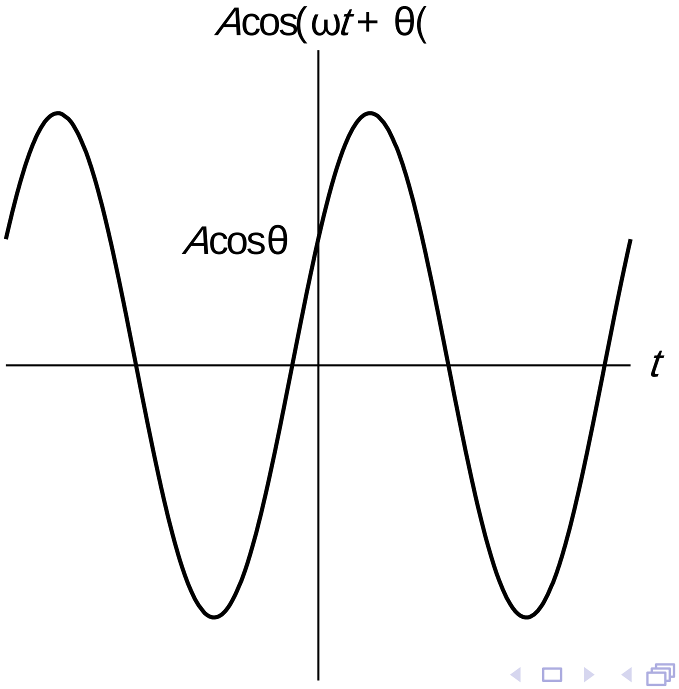
Section 2.3

# Elementary Signals

- A (CT) real sinusoid is a function of the form

$$x(t) = A\cos(\omega t + \theta),$$

where $A$, $\omega$, and $\theta$ are *real* constants.
- Such a function is periodic with *fundamental period* $T = \frac{2\pi}{|\omega|}$ and *fundamental frequency* $|\omega|$.
- A real sinusoid has a plot resembling that shown below.



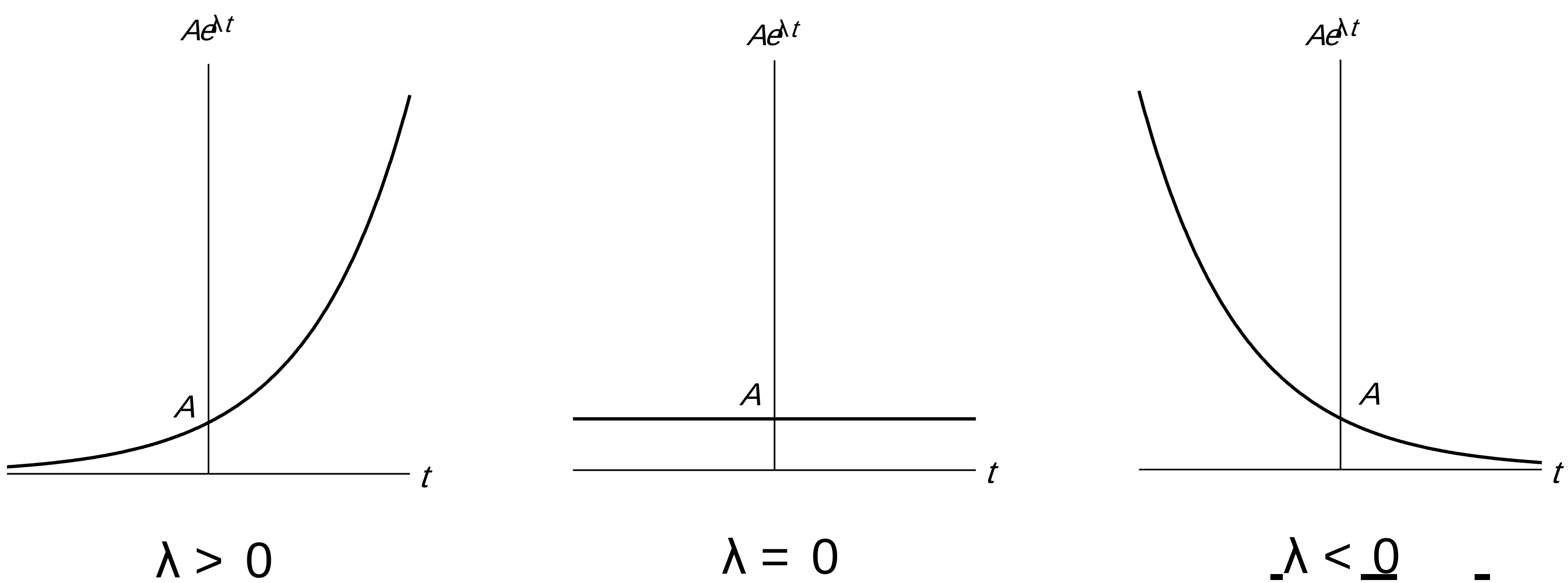
- A (CT) complex exponential is a function of the form

$$x(t) = Ae^{\lambda t},$$

where $A$ and $\lambda$ are *complex* constants.

- A complex exponential can exhibit one of a number of *distinct modes of behavior*, depending on the values of its parameters $A$ and $\lambda$.
- For example, as special cases, complex exponentials include real exponentials and complex sinusoids.

- A real exponential is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where $A$ and $\lambda$ are restricted to be *real* numbers.
- A real exponential can exhibit one of *three distinct modes* of behavior, depending on the value of $\lambda$, as illustrated below.
- If $\lambda > 0$, $x(t)$ *increases* exponentially as $t$ increases (i.e., a growing exponential). If
- $\lambda < 0$, $x(t)$ *decreases* exponentially as $t$ increases (i.e., a decaying exponential). If $\lambda$
- $= 0$, $x(t)$ simply equals the *constant* $A$.



- A complex sinusoid is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where $A$ is *complex* and $\lambda$ is *purely imaginary* (i.e., $\operatorname{Re}\{\lambda\} = 0$).
- That is, a (CT) complex sinusoid is a function of the form

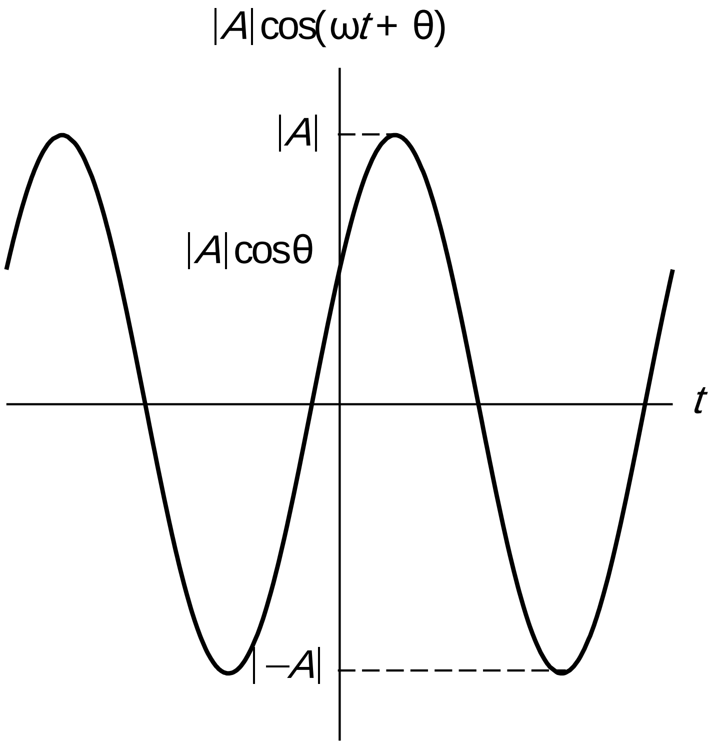
$$x(t) = Ae^{j\omega t},$$

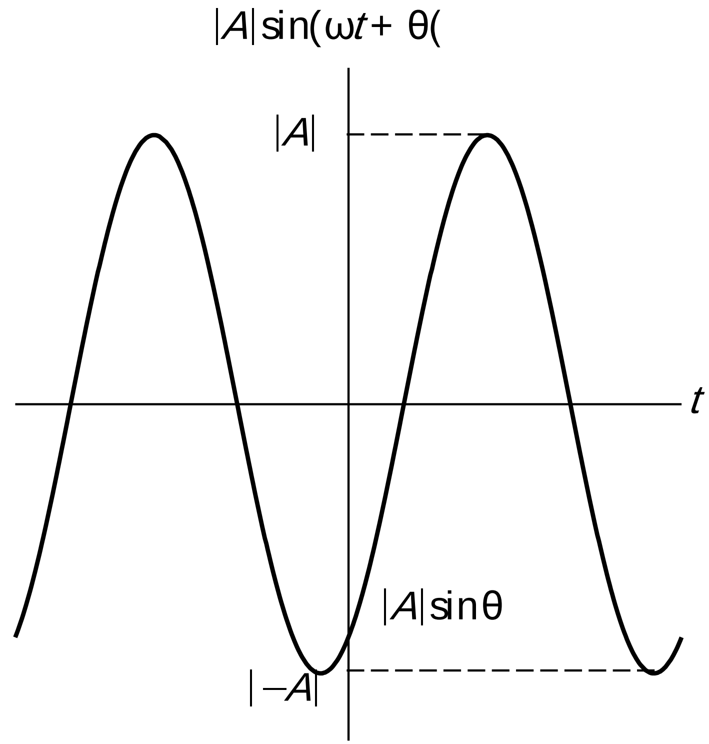
where $A$ is *complex* and $\omega$ is *real*.
- By expressing $A$ in polar form as $A = |A|e^{j\theta}$ (where $\theta$ is real) and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A|\cos(\omega t + \theta)}_{\operatorname{Re}\{x(t)\}} + j\underbrace{|A|\sin(\omega t + \theta)}_{\operatorname{Im}\{x(t)\}}.$$

- Thus, $\operatorname{Re}\{x\}$ and $\operatorname{Im}\{x\}$ are the same except for a time shift.
- Also, $x$ is periodic with *fundamental period* $T = \frac{2\pi}{|\omega|}$ and *fundamental frequency* $|\omega|$.

- The graphs of $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ have the forms shown below.



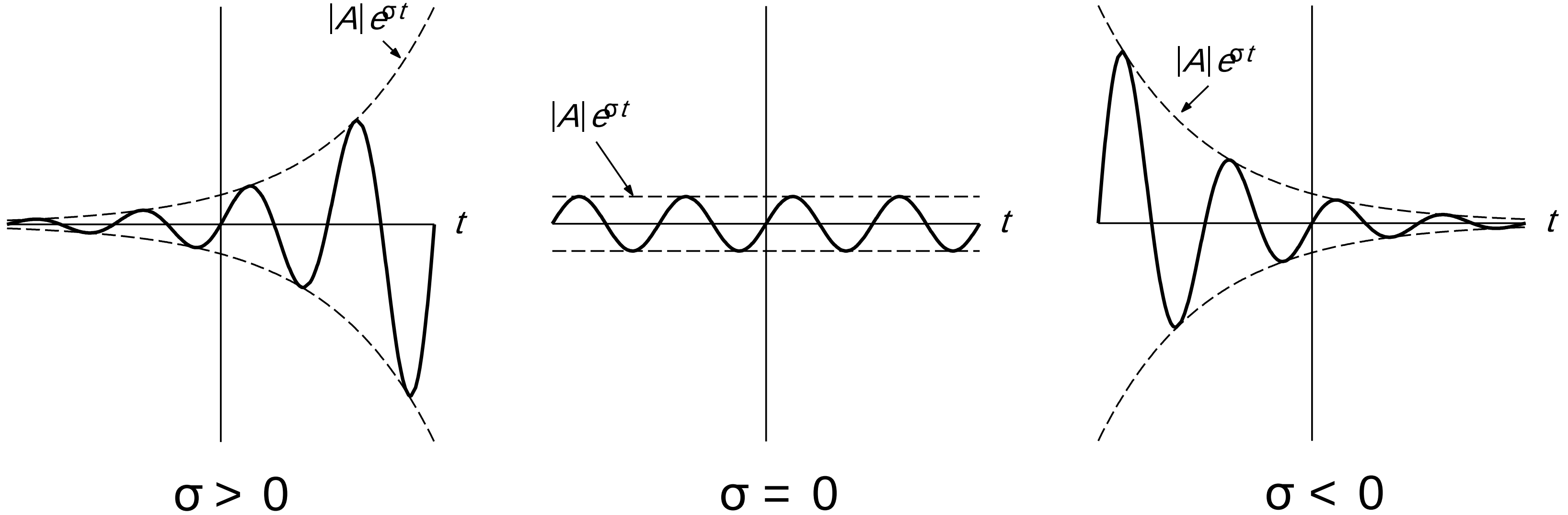



- In the most general case of a complex exponential $x(t) = Ae^{\lambda t}$, $A$ and $\lambda$ are both *complex*.
- Letting $A = |A| e^{j\theta}$ and $\lambda = \sigma + j\omega$ (where $\theta$, $\sigma$, and $\omega$ are real), and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A| e^{\sigma t} \cos(\omega t + \theta)}_{\operatorname{Re}\{x(t)\}} + j \underbrace{|A| e^{\sigma t} \sin(\omega t + \theta)}_{\operatorname{Im}\{x(t)\}}.$$

- Thus, $\operatorname{Re}\{x\}$ and $\operatorname{Im}\{x\}$ are each the product of a real exponential and real sinusoid.
- One of *three distinct modes* of behavior is exhibited by $x(t)$, depending on the value of $\sigma$.
- If $\sigma = 0$, $\operatorname{Re}\{x\}$ and $\operatorname{Im}\{x\}$ are *real sinusoids*.
- If $\sigma > 0$, $\operatorname{Re}\{x\}$ and $\operatorname{Im}\{x\}$ are each the *product of a real sinusoid and a growing real exponential*.
- If $\sigma < 0$, $\operatorname{Re}\{x\}$ and $\operatorname{Im}\{x\}$ are each the *product of a real sinusoid and a decaying real exponential*.

- The *three modes of behavior* for $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are illustrated below.



## Relationship Between Complex Exponentials and Real

- From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as

$$Ae^{j\omega t} = A\cos\omega t + jA\sin\omega t.$$

- Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities

$$A\cos(\omega t+\theta) = \frac{A}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] \quad \text{and}$$

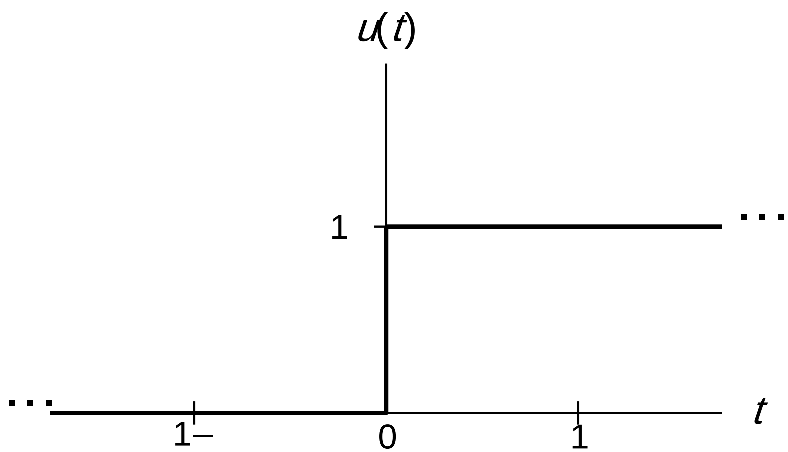
$$A\sin(\omega t+\theta) = \frac{A}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right].$$

- Note that, above, we are simply *restating results* from the (appendix) material on complex analysis.

- The unit-step function (also known as the Heaviside function), denoted $u$, is defined as

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

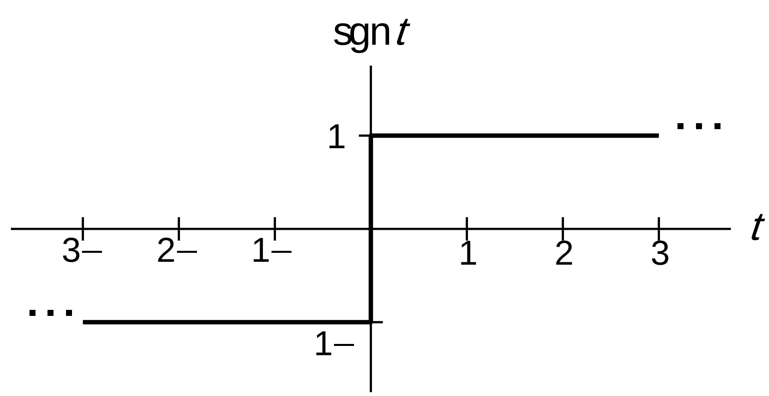
- Due to the manner in which $u$ is used in practice, the actual *value of $u(0)$* is unimportant. Sometimes values of $0$ and $\frac{1}{2}$ are also used for $u(0)$.
- A plot of this function is shown below.



- The signum function, denoted $\operatorname{sgn}$, is defined as

$$\operatorname{sgn} t = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0. \end{cases}$$

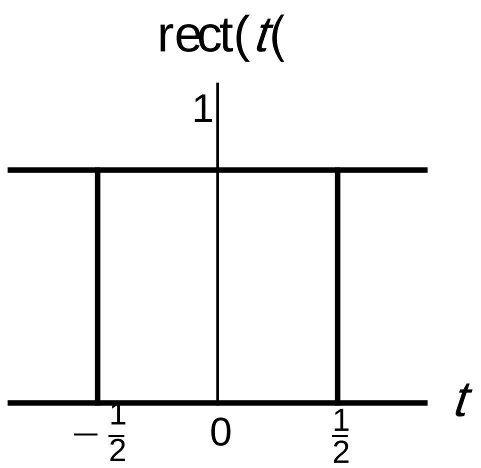
- From its definition, one can see that the signum function simply computes the *sign* of a number.
- A plot of this function is shown below.



- The rectangular function (also called the unit-rectangular pulse function), denoted rect, is given by

$$\operatorname{rect}(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

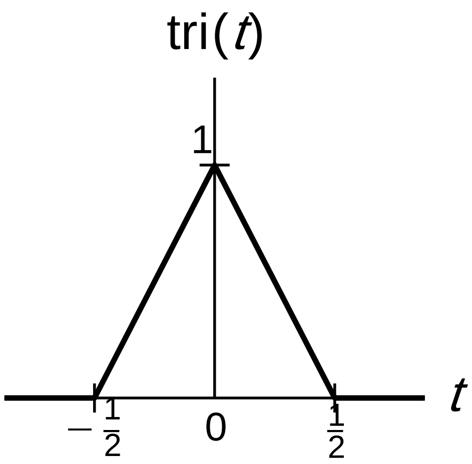
- Due to the manner in which the rect function is used in practice, the actual *value of* $\operatorname{rect}(t)$ *at* $t = \pm\frac{1}{2}$ is unimportant. Sometimes different values are used from those specified above.
- A plot of this function is shown below.



- The triangular function (also called the unit-triangular pulse function), denoted tri, is defined as

$$\operatorname{tri}(t) = \begin{cases} 1 - 2|t| & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

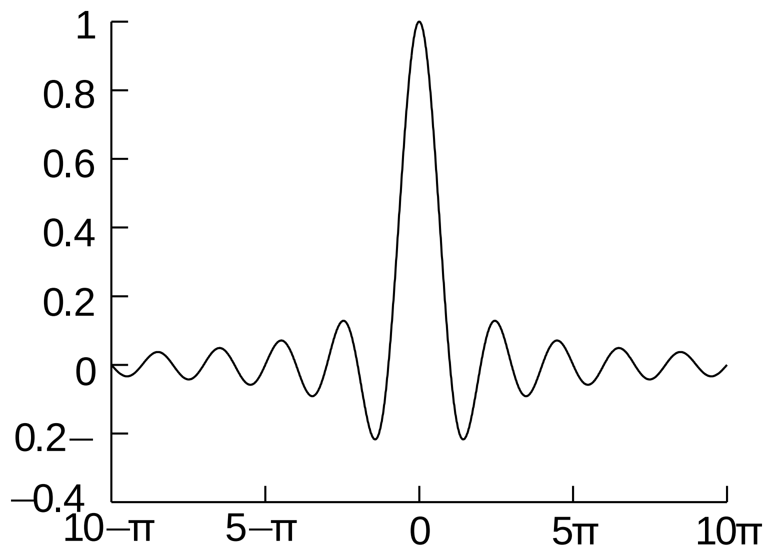
- A plot of this function is shown below.



- The cardinal sine function, denoted $\mathrm{sinc}$, is given by

$$\mathrm{sinc}(t) = \frac{\sin t}{t}.$$

- By l'Hopital's rule, $\mathrm{sinc}\,0 = .1$
- A plot of this function for part of the real line is shown below. [Note that the oscillations in $\mathrm{sinc}(t)$ do not die out for finite $t$[.



- The unit-impulse function (also known as the Dirac delta function or delta function), denoted $\delta$, is defined by the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and}$$
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

- Technically, $\delta$ is not a function in the ordinary sense. Rather, it is what is known as a *generalized function*. Consequently, the $\delta$ function sometimes behaves in unusual ways.
- Graphically, the delta function is represented as shown below.